Available online at www.sciencedirect.com
SCIENCE DIRECT®

Journal of Sound and Vibration 292 (2006) 1011–1014

JOURNAL OF SOUND AND VIBRATION

www.elsevier.com/locate/jsvi

Short Communication

Random vibrations of swings

Daniil V. Iourtchenko*

MIIT, Department of Applied Mathematics, Moscow 127994, Russian Federation

Received 20 July 2005; received in revised form 20 July 2005; accepted 6 September 2005
Available online 2 November 2005

Abstract

The paper considers a problem of random vibrations of swings—a pendulum with variable length. The goal of the paper is to estimate mean response energy of the system, subjected to external Gaussian white noise. The Energy Balance method is used to derive analytical results.

1. Introduction

The paper considers a stochastic analysis of a pendulum with instantaneous, stepwise variations of its length, subjected to an external random excitation. This system is also known as *swings* and it may be regarded as a strongly nonlinear one. Analysis of this system cannot be performed by known stochastic averaging procedures since certain conditions for their implementation are violated [1,2]. On the other hand, stochastic analysis may be conducted by the Energy Balance method [3] developed for piecewise-conservative systems [3]. The method, basically, separates the system's motion into two parts: a motion between switches and energy losses due to switches. This procedure results in mean value of the system response energy as a function of a time interval between two consecutive switches—mean quarter period of the system. This formula is exact and therefore does not require small changes of response energy within a period.

2. Analytical analysis

Consider a mathematical pendulum with variable length subjected to standard, zero mean Gaussian white noise:

$$\frac{\mathrm{d}}{\mathrm{d}t}(L^2\dot{\phi}) + \Omega^2 L\phi = \xi(t), \quad 0 \leqslant t \leqslant T,$$

$$L = (1 + R\,\mathrm{sign}(\phi\dot{\phi})), \quad \Omega^2 = g/L_0, \quad 0 < R < 1,$$

*Corresponding author: Tel./fax: +095 684 2309.
E-mail address: daniil@miami.edu.

doi:10.1016/j.jsv.2005.09.005

$$\langle \xi(t), \xi(t+\tau) \rangle = D\delta(\tau). \tag{1}$$

where D is noise intensity. Similar system with random excitation applied to the pendulum's suspension point has been analyzed in Ref. [4]. It has been shown that such length variations lead to energy losses in Eq. (1) with no noise. The Energy Balance has been used in Ref. [4] to obtain an analytical expression for mean system response energy, which proved to be more accurate than its asymptotic expression for limiting case of $R \to 0$.

Introducing a new variable p one may rewrite Eq. (1) and its energy between switches as

$$\dot{\phi} = \frac{p}{(1 + R\,\mathrm{sign}(\phi p))^2}, \quad \dot{p} = -\Omega^2(1 + R\,\mathrm{sign}(\phi p))\phi + \xi(t),$$

$$H = \frac{1}{2}\left\{\frac{p^2}{(1 + R\,\mathrm{sign}(\phi p))^2} + \Omega^2(1 + R\,\mathrm{sign}(\phi p))\phi\right\},$$

$$\dot{H} = \frac{p\xi}{(1 + R\,\mathrm{sign}(\phi p))^2}. \tag{2}$$

The later expression should be understood in the Stratonovich sense. To evaluate its mean, one has to add the Wong-Zakai correction term, so that a value of mean energy between two consecutive switches is

$$\overline{\dot{H}} = \frac{D/2}{(1 + R\,\mathrm{sign}(\phi p))^2}, \quad \overline{H}(t) = \frac{Dt/2}{(1 + R\,\mathrm{sign}(\phi p))^2} + H(0). \tag{3}$$

Expression (3) indicates that energy between switches grows linearly in time. It is important to stress that the influence of noise intensity is also different—it is large when ϕ and p have different sign and smaller otherwise.

Assume, without loss of generality, that a motion of the system at the current cycle starts to the right from the system's equilibrium position, so that $\phi > 0$ and $p > 0$. Then the energy evolution between switches is (the bar over mean energy is skipped for brevity):

$$H(t_{1/4} - 0) = \frac{Dt_{1/4+}/2}{L_+^2} + H(0),$$

$$H(t_{1/2} - 0) = \frac{Dt_{1/4-}/2}{L_-^2} + H(t_{1/4} + 0), \quad L_\pm = (1 \pm R). \tag{4}$$

where $t_{1/4\pm}$ is random time between two consecutive switches, with the sign corresponding to that in front of R, $H(t_{1/4} - 0)$ and $H(t_{1/2} - 0)$ values of energy just before switches after the first and second quarter periods and $H(t_{1/4} + 0)$ is a value of energy right after the first switch. Next, let us evaluate energy losses due to switches. At the maximum system's displacement its velocity is zero and therefore from Eq. (3) one obtains

$$H(t_{1/4} + 0) = H(t_{1/4} - 0)\frac{L_-}{L_+}. \tag{5}$$

At the system's equilibrium position, system's displacement is zero and according to the conservation of angular momentum law the energy losses are:

$$H(t_{1/2} + 0) = H(t_{1/2} - 0)\frac{L_-^2}{L_+^2}. \tag{6}$$

Combining expressions (6), (5) and (4), applying unconditional averaging and imposing the condition of stationary response, i.e., $\langle H(t_{1/2} + 0)\rangle = \langle H(0)\rangle$ results in

$$\langle H(0)\rangle = \frac{L_+}{L_+^3 - L_-^3}\left(\frac{L_-^3}{L_+^3}\frac{DT_{1/4+}}{2} + \frac{DT_{1/4-}}{2}\right), \tag{7}$$

where $T = \langle t \rangle$ mean time between switches. Assuming that the value of mean quarter periods of systems (1) are equal to

$$T_{1/4\pm} = (\pi/2\Omega)\sqrt{1 \pm R}, \tag{8}$$

expression (7) may be simplified to

$$\langle H(0)\rangle = \sigma^2 \frac{L_+/2}{1+R^2/3}\left(\frac{L_-^3}{L_+^3}\sqrt{(1+R)}+\sqrt{(1-R)}\right),$$

$$\sigma^2 = \frac{D\pi}{12\Omega R}. \tag{9}$$

It should be noted that the same value of σ^2 has been obtained in Ref. [4]. To obtain the overall value of the response energy let us average expression (3) within the half a period:

$$\langle H\rangle = \frac{1}{T}\int_0^T \overline{H}(t)\,\mathrm{d}t$$
$$= \langle H(0)\rangle \frac{T_{1/4+}}{T_{1/2}} + H_1 \frac{T_{1/4-}}{T_{1/2}} + \frac{D}{4T_{1/2}}\left(\frac{T_{1/4+}^2}{L_+^2}+\frac{T_{1/4-}^2}{L_-^2}\right), \tag{10}$$

$$H_1 = \sigma^2\gamma \frac{L_-/2}{1+R^2/3}, \quad \gamma = \sqrt{1+R}+\sqrt{1-R}.$$

In the case of assumption (8) expression (10) simplified to

$$\langle H\rangle = \sigma^2 \Psi(R),$$

$$\Psi(R) = \frac{(1-R^2)^{3/2}+(3R^2+1)}{\gamma(1+R^2/3)(1-R^2)}, \tag{11}$$

so that in the limiting case of small R:

$$\lim_{R\to 0}\Psi(R) \to 1, \quad \langle H\rangle \approx \sigma^2. \tag{12}$$

Unfortunately expression (11) seems to provide wrong estimates, going to infinity, for R approaching unity. On the other hand, asymptotic result (12) should provide accurate estimations for small values of R.

3. Discussion of results

In Fig. 1 results of Monte-Carlo (MC) simulation are compared with asymptotic result (12). It is seen that the results of numerical simulation agree very well with asymptotic analytical expression even for large values of R. Therefore, contrary to the case considered in Ref. [4], where the asymptotic value was less accurate than that obtained by the Energy Balance method, here the asymptotic result proved to be very accurate within the

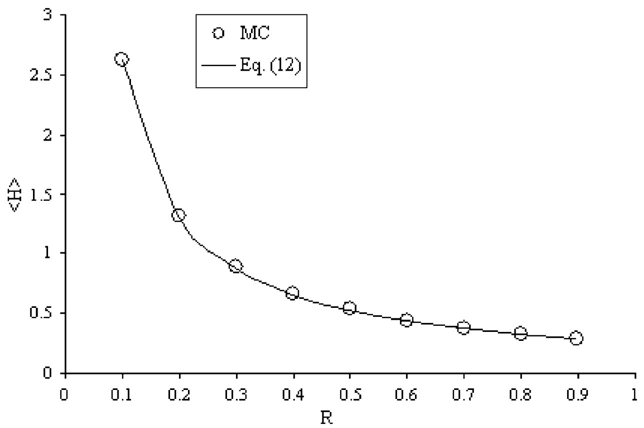

Fig. 1. Mean response energy vs. R for $D = 1$ calculated analytically (Eq. (12)) 'o' and numerically (MC) '—'.

whole range of R values and even large values of noise intensity D. Concluding, it must be emphasized that mean response energy behavior is well described by expression (12).

References

[1] M.F. Dimentberg, *Statistical Dynamics of Nonlinear and Time-Varying Systems*, Research Studies Press, Taunton, UK, 1988.

[2] R.Z. Khasminskii, A limit theorem for solution of differential equations with random right-hand-side, *Journal of Probability and Application* 11 (3) (1966) 390–405.

[3] D.V. Iourtchenko, M.F. Dimentberg, Energy Balance for random vibration of piecewise-conservative systems, *Journal of Sound and Vibration* 248 (5) (2001) 913–923.

[4] M.F. Dimentberg, On the theory of swings, *Journal of Vibration and Control* 8 (2002) 311–319.